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Analytical solution to the flow between two coaxial rotating disks using HAM.

Abhijit Das*

Department of Mathematics, NIT Rourkela, Rourkela-769008, India

Abstract

The steady flow of a viscous, incompressible fluid between two infinite rotating coaxial disks is considered in this paper. An exact analytical solution to the problem is given by an effective analytical method called Homotopy Analysis Method (HAM).Three different cases such as one disk rotating with constant angular velocity while the other one is at rest, disks rotating in same as well as opposite sense with different angular velocities are considered and discussed for small Reynolds number.

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Keywords: Rotating system; Similarity Transformations; Reynolds number; HAM.

1. Introduction

The flow between two coaxial rotating disks has attracted many researchers due to its theoretical as well as practical importance.Kàrmàn [10] was the first one to consider and solve the steady, viscous, incompressible flow over a rotating disk. Later different researchers such as Batchelor [2], Stewartson [19], Rogers and Lance [11], Pearson [14] and so on solved different generalized versions of this problem numerically, theoretically or experimentally.Sriwastava [18] discussed the steady flow between two coaxial rotating disks for Reiner-Rivlin fluid at small Reynolds number, the unsteady case for the same problem for Newtonian fluid were studied by Greenspan [7], Howard and Greenspan [8], the non-Newtonian fluid flow problem of the same type were studied by Bhatnagar [3], Bhatnagar and Rajeswari [4] for small Reynolds number. Yesilata [22] investigated numerically the effect of viscous dissipation for the viscoelastic flow between two rotating parallel discs. A systematic study on the qualitative nature of flow structure between two rotating co-axial disks at a relatively wide range of rotational conditions was studied experimentally by Soong et al. [20]. PECheux et al. [15] studied about the stability analysis of the flow between two coaxial disks experimentally and numerically. Nazir et al. [13] discussed numerically the effects of disks contracting, rotation and heat transfer on the viscous fluid between heated contracting rotating disks. Hatami et al. [9] investigated asymmetric laminar flow and heat transfer of nanofluid between contracting rotating disks using least square method. However, to the best of my knowledge these results are either numerical or analytical-numerical.

* Abhijit Das. Tel.: +91-8093329659.
E-mail address: 513ma1001@nitrkl.ac.in

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In present investigation, the steady, viscous, incompressible flow between two rotating coaxial disks is considered and homotopy analysis method devised by Liao [12] is adopted to give an exact analytical solution to the problem. The homotopy analysis method has been successfully applied to various nonlinear problems. Yang and Liao [21] gave an exact analytical solution to the steady von Kàrmàn flow using algebraic-exponential base functions, Sajid and Hayat [17] gave analytical solution to the MHD viscous flow due to a stretching sheet, Rashidi et al.[16] produced analytical solution for the flow over a rotating disk in porous medium with heat transfer, Abbasbandy et al. [1] used HAM to produce approximate solution of fractional integro-differential equations, Aziz and Nabil [6] investigated the time-dependent heat source/sink on heat transfer characteristics of the unsteady mixed convection flow over an exponentially stretching surface analytically using HAM. All these example shows the validity of homotopy analysis method and so in this paper we employ this method for the classical viscous flow between two coaxial rotating disks.

2. Formulation of the Problem

Let u, v, w be the velocity components in the direction of increase of cylindrical polar coordinates r, θ, z respectively. Let the lower disk placed at $z = 0$ rotates with an angular velocity $s\Omega$ and the upper disk placed at $z = d$ rotates with an angular velocity Ω. Using similarity transformations [5]:

$$u = -\frac{1}{2}r\Omega H'(\eta), v = r\Omega G(\eta), w = d\Omega H(\eta) \tag{1}$$

where,
$\eta = \frac{z}{d}$ and dash denotes differentiation w.r.t η, the momentum equations reduce to (equation of continuity is identically satisfied):

$$H^{iv} - R(HH''' + 4GG') = 0 \tag{2}$$

$$G'' + R(H'G - HG') = 0 \tag{3}$$

With the boundary conditions:

$$G(0) = s, H(0) = 0, H'(0) = 0, G(1) = 1, H(1) = 0, H'(1) = 0 \tag{4}$$

where,
$R = \frac{\rho\Omega d^2}{\mu}$ is the Reynolds number.

3. HAM Solution

In the frame of HAM and due to the boundary conditions (4) we choose the base function $\{\eta^m | m \geq 0\}$ to express $H(\eta)$and $G(\eta)$. The initial approximations are chosen as:

$$H_0(\eta) = 0, \tag{5}$$

$$G_0(\eta) = s + (1 - s)\eta. \tag{6}$$

The auxilary linear operators $L_H(f)$ and $L_G(f)$ are defined as:

$$L_H(f) = H^{iv}, \tag{7}$$

$$L_G(f) = G'' \tag{8}$$

with the following properties

$$L_H(c_1 + c_2\eta + c_3\eta^2 + c_4\eta^3) = 0 \tag{9}$$

$$L_G(c_5 + c_6\eta) = 0 \tag{10}$$

where,
$c_i, i = 1 - 6$ are arbitrary constants. Now we construct the zeroth-order deformation equations:

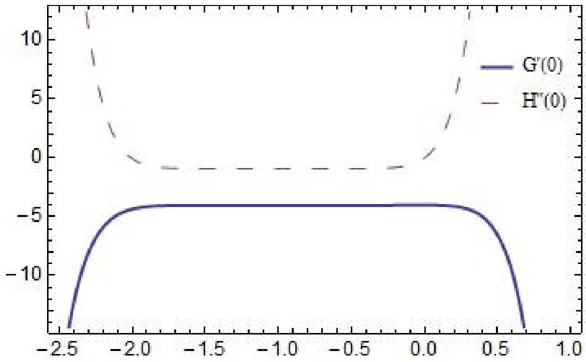

Fig. 1. $\hbar$-curve for $H''(0), G'(0)$for $R = 0.2, s = 5$ obtained by 10^{th} order HAM approximation.

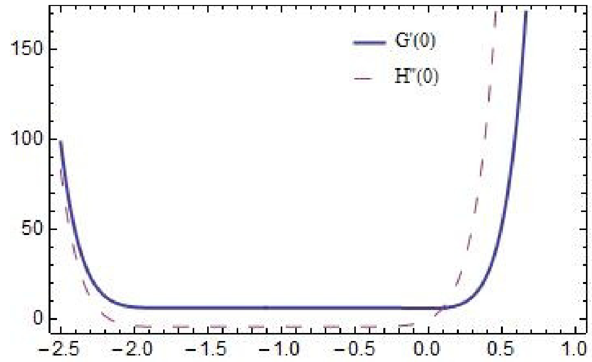

Fig. 2. $\hbar$-curve for $H''(0), G'(0)$for $R = 0.8, s = -5$ obtained by 10^{th} order HAM approximation.

$$(1-q)L_H[\hat{H}(\eta;q) - H_0(\eta)] = q\hbar_H N_H[\hat{H}(\eta;q), \hat{G}(\eta;q)] \tag{11}$$

$$(1-q)L_G[\hat{G}(\eta;q) - G_0(\eta)] = q\hbar_G N_G[\hat{H}(\eta;q), \hat{G}(\eta;q)] \tag{12}$$

subject to the boundary conditions:

$$\hat{G}(0;q) = \hat{H}(0;q) = \left.\frac{\partial \hat{H}(\eta;q)}{\partial \eta}\right|_{\eta=0} = 0, \hat{G}(1;q) = \hat{H}(1;q) = \left.\frac{\partial \hat{H}(\eta;q)}{\partial \eta}\right|_{\eta=1} = 0 \tag{13}$$

Where, N_H and N_G are two differential operators defined by:

$$N_H = \frac{\partial^4 \hat{H}(\eta;q)}{\partial \eta^4} - R\left(\hat{H}(\eta;q)\frac{\partial^3 \hat{H}(\eta;q)}{\partial \eta^3} + 4\hat{G}(\eta;q)\frac{\partial \hat{G}(\eta;q)}{\partial \eta}\right) \tag{14}$$

$$N_G = \frac{\partial^2 \hat{G}(\eta;q)}{\partial \eta^2} + R\left(\frac{\partial \hat{H}(\eta;q)}{\partial \eta}\hat{G}(\eta;q) - \hat{H}(\eta;q)\frac{\partial \hat{G}(\eta;q)}{\partial \eta}\right) \tag{15}$$

where, $q \in [0,1]$ is the embedding parameter,$\hbar_H$ and $\hbar_G$ are auxiliary nonzero parameters. When q varies from 0 to 1 $\hat{H}(\eta;q)$ and $\hat{G}(\eta;q)$ varies from $H_0(\eta), G_0(\eta)$ to $H(\eta), G(\eta)$.
Therefore by Taylor's theorem we have

$$\hat{H}(\eta;q) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta)q^m \tag{16}$$

$$\hat{G}(\eta;q) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta)q^m \tag{17}$$

where,
$H_m(\eta) = \frac{1}{m!}\frac{\partial \hat{H}(\eta;q)}{\partial \eta}\Big|_{q=0}, G_m(\eta) = \frac{1}{m!}\frac{\partial \hat{G}(\eta;q)}{\partial \eta}\Big|_{q=0}$. Suppose that the auxiliary parameters are so selected that the series (16) and (17) are convergent at $q = 1$, then we can write

$$H(\eta) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta) \tag{18}$$

$$G(\eta) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta) \tag{19}$$

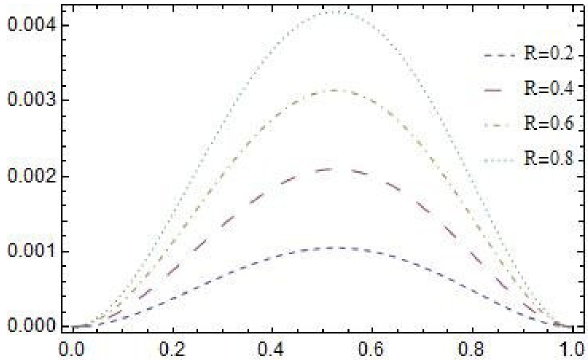

Fig. 3. The profile of $H(\eta)$ when $s = 0$ obtained by 10^{th} order HAM approximation.

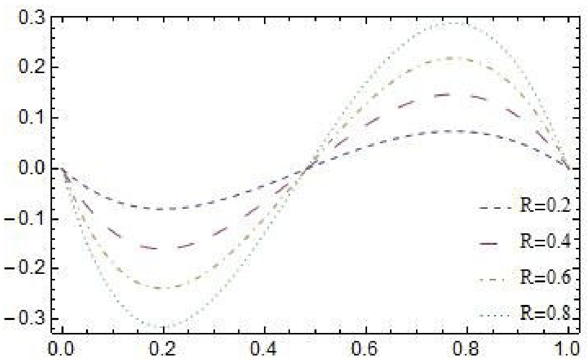

Fig. 4. The profile of $H'(\eta)$ when $s = 5$ obtained by 10^{th} order HAM approximation.

Differentiating m times the zeroth-order differential equations (11), (12) then dividing by $m!$ and then setting $q = 0$ we get the following m^{th} order deformation equations

$$L_H[H_m(\eta) - \chi_m H_{m-1}(\eta)] = \hbar_H R_m^H \tag{20}$$

$$L_G[G_m(\eta) - \chi_m G_{m-1}(\eta)] = \hbar_G R_m^G \tag{21}$$

Subject to the boundary conditions

$$H_m(0) = G_m(0) = H_m'(0) = H_m(1) = G_m(1) = H_m'(1) = 0 \tag{22}$$

where,

$$R_m^H = \frac{\partial^4 H_{m-1}(\eta)}{\partial \eta^4} - R\sum_{n=0}^{m-1}\left(H_n(\eta)\frac{\partial^3 H_{m-1-n}(\eta)}{\partial \eta^3} + 4G_n(\eta)\frac{\partial G_{m-1-n}(\eta)}{\partial \eta}\right) \tag{23}$$

$$R_m^G = \frac{\partial^2 G_{m-1}(\eta)}{\partial \eta^2} + R\sum_{n=0}^{m-1}\left(\frac{\partial H_n(\eta)}{\partial \eta}G_{m-1-n}(\eta) - H_n(\eta)\frac{\partial G_{m-1-n}(\eta; q)}{\partial \eta}\right) \tag{24}$$

And,

$$\chi_m = \begin{cases} 1 & m > 1 \\ 0 & m \leq 0 \end{cases} \tag{25}$$

Finally we use the symbolic software MATHEMATICA to solve the linear equations (20)-(22) and we get,

$$\begin{aligned} H_1(\eta) = &-\frac{1}{15}\hbar R\eta^2 - \frac{1}{30}\hbar Rs\eta^2 + \frac{1}{10}\hbar Rs^2\eta^2 + \frac{1}{10}\hbar R\eta^3 + \frac{2}{15}\hbar Rs\eta^3 - \frac{7}{30}\hbar Rs^2\eta^3 \\ &-\frac{1}{6}\hbar Rs\eta^4 + \frac{1}{6}\hbar Rs^2\eta^4 - \frac{1}{30}\hbar R\eta^5 + \frac{1}{15}\hbar Rs\eta^5 - \frac{1}{30}\hbar Rs^2\eta^5, \\ &G_1(\eta) = 0 \end{aligned} \tag{26}$$

and so on. Using equations (16),(17) and higher order solutions of equations (20)-(22) we can express the solutions as

$$H(\eta) = \sum_0^\infty a_k\eta^k \tag{27}$$

$$G(\eta) = \sum_0^\infty b_k\eta^k \tag{28}$$

Where the coefficients can be determined from the higher order solutions given by equations (20)-(22).Also the convergence of the analytical solution mainly depends upon the parameter $\hbar$ and the proper value of $\hbar$ is chosen from the $\hbar$-curves.

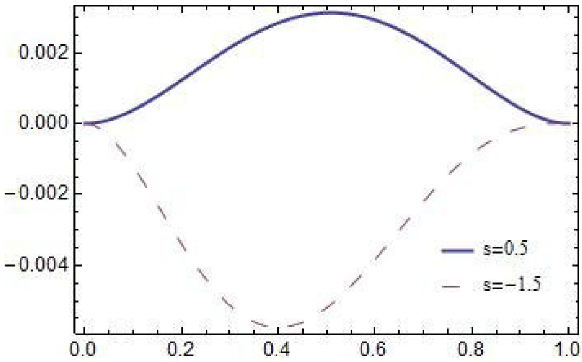

Fig. 5. The profile of $H(\eta)$ when $R = 0.8, s = 0.5, s = -1.5$ obtained by 10^{th} order HAM approximation.

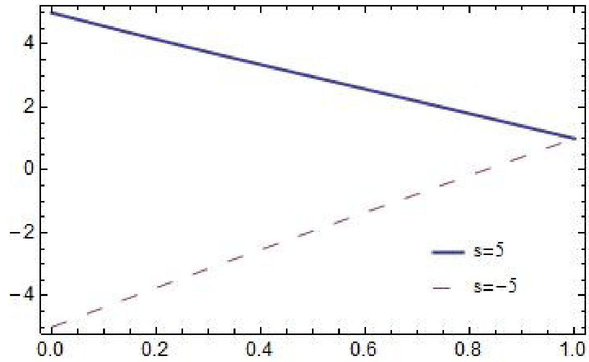

Fig. 6. The profile of $G(\eta)$when $R = 0.8, s = 5, s = -5$ obtained by 10^{th} order HAM approximation.

4. Results and Discussions

To demonstrate the efficiency and accuracy of HAM for the above problem, the results are represented graphically. For the case when lower disc is held at rest ($s = 0$), Fig. 3 represents the curves of the function $H(\eta)$ representing the axial velocity for different values of the Reynolds number, it can be observed that the profiles are nearly parabolic and always positive. The case when the lower disc rotates faster than the upper one ($s = 5$) in same sense can be studied from Fig. 4 representing the curves of the function $H'(\eta)$ and it can be noted that for different values of Reynolds number $H'(\eta)$ is negative in the lower half and positive in the upper half. Comparison between the cases when the disks rotate in opposite sense with different angular velocities and the disks rotate in same sense can be studied from Fig. 5 and 6. From these figures, the present analytical results seem to be in good agreement with the results obtained by Bhatnagar[5] for viscous case.

5. Conclusion

In this paper, homotopy analysis method (HAM) was adopted to give an exact analytical solution for the steady, viscous, incompressible flow between two coaxial rotating disks. Homotopy analysis method does not depend on any small or large physical parameters and thus is valid for both weakly as well as strongly nonlinear problems. Unlike perturbation techniques, HAM provides us an easy way to control and adjust the convergence of series solution by means of auxiliary parameter $\hbar$, also known as convergence control parameter. Thus the convergence control parameter was determined using $\hbar$-curves (two curves are shown in Fig. 1 and 2) and our analytical solutions are shown graphically (in Fig. 3-6). It is to be noted that our solution is analytic i.e. the form of the solution is known and no numerical technique is needed to get the coefficients. This verifies that HAM is an effective and useful tool and hence can be applied to solve more complicated and nonlinear problems arising in science and engineering.

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